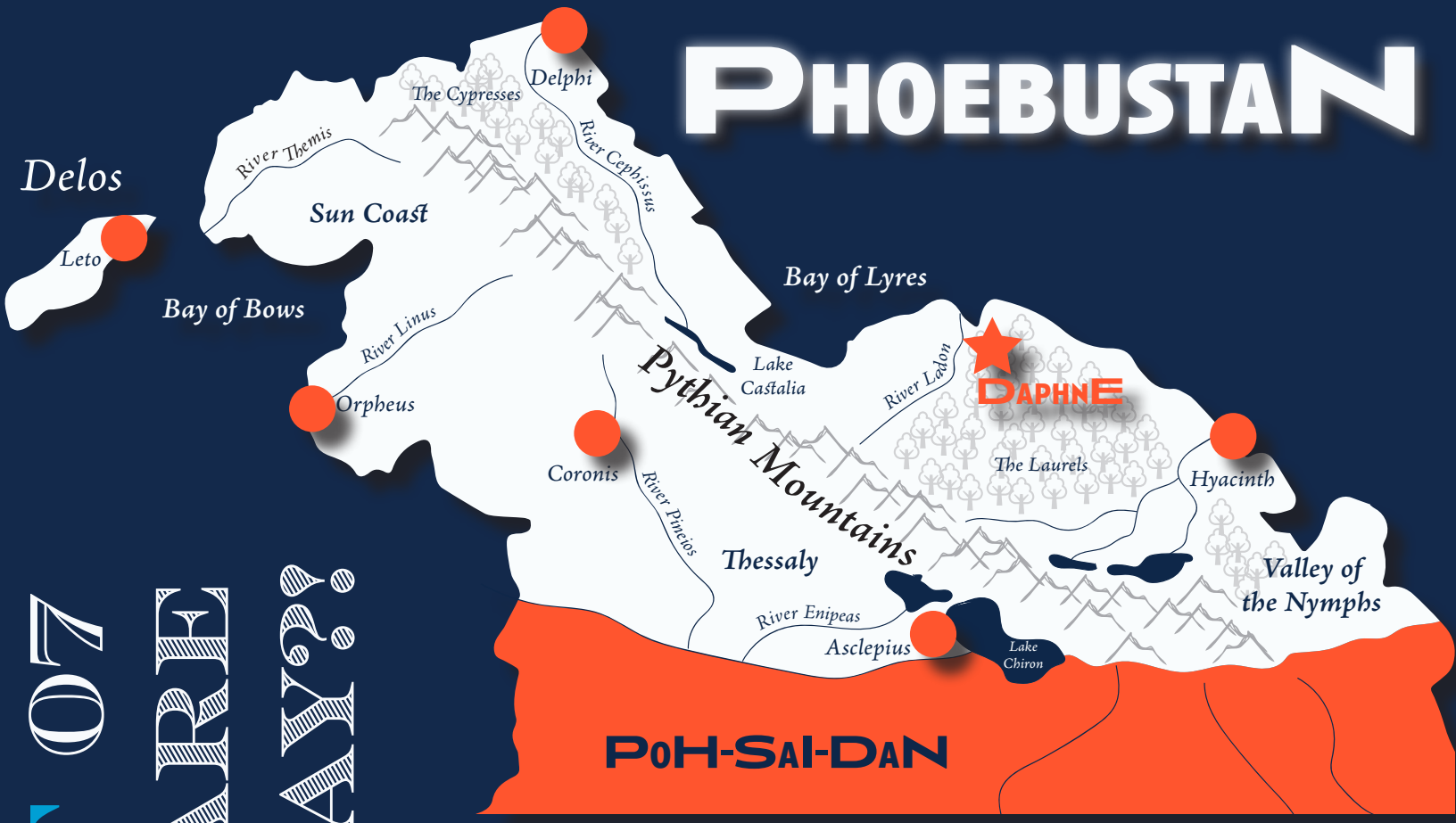
PROBLEM SET 07

WHOSE SIDE ARE YOU ON, ANYWAY?



SOCRATES

Χαίρε, friend. I haven't been feeling my best lately, so I've made the long trek to Asclepius, Phœbustan, where the Lake Chiron Medical Center stands as the paragon of all things healthy. I probably should have come here when I was feeling well...

Anyway, I've been sitting in this waiting room for what feels like forever, and they're playing gods-awful cable news again. [*Remember when that seemed like our biggest worry?! How far we have come.*] It's unbearable, but my Mílo Music app doesn't want to work with the available wifi. What fresh Hades is this, medical center? Isn't it enough that you're about to poke and prod me like an animal?!

Nevertheless, I feel like I've learned something on these cable news. [*I must have a fever to be saying such things.*] I've found that Phœbustani democracy creates many interesting opportunities for coalition-building and that these coalitions can often create new and unexpected strategic tensions. What's more, the various issues at stake in Phœbustan seem to generate particular kinds of coalitions, hence particular strategic tensions! And then these poor cable news pundits have to pretend like they know how to make predictions of what will happen with all these issues, their subsequent coalitions, and the subsequent strategic tensions! Clearly, these idiotic talking heads are not cut out for a job this difficult.

But we are, don't you think? Let's find out.

You will have noticed from the map that there are seven primary cities in Phœbustan: the capital of Daphne, Coronis, Asclepius, Delphi, Hyacinth, Leto, and Orpheus. Traditionally these cities have fallen into three natural regions: the Sun Coast [including Leto and Orpheus], Thessaly [including Coronis and Asclepius], and Laurelia [including Daphne, Delphi, and Hyacinth]. These regions have rough equal populations, and Phœbustani politics often revolve around them.

Naturally, exceptions exist. On some matters it's easiest to think about the seven cities as independent units, which makes things a bit confusing. The easiest partitioning scheme is the one that pits the capital of Delphi against the other six cities.

In other words, we've got a bit of a mess on our hands, but surely we can handle it. Maybe. No, definitely. Probably. *Shut up.*

SEVEN OR THREE OR TWO

PART 1 | THE TAXMAN

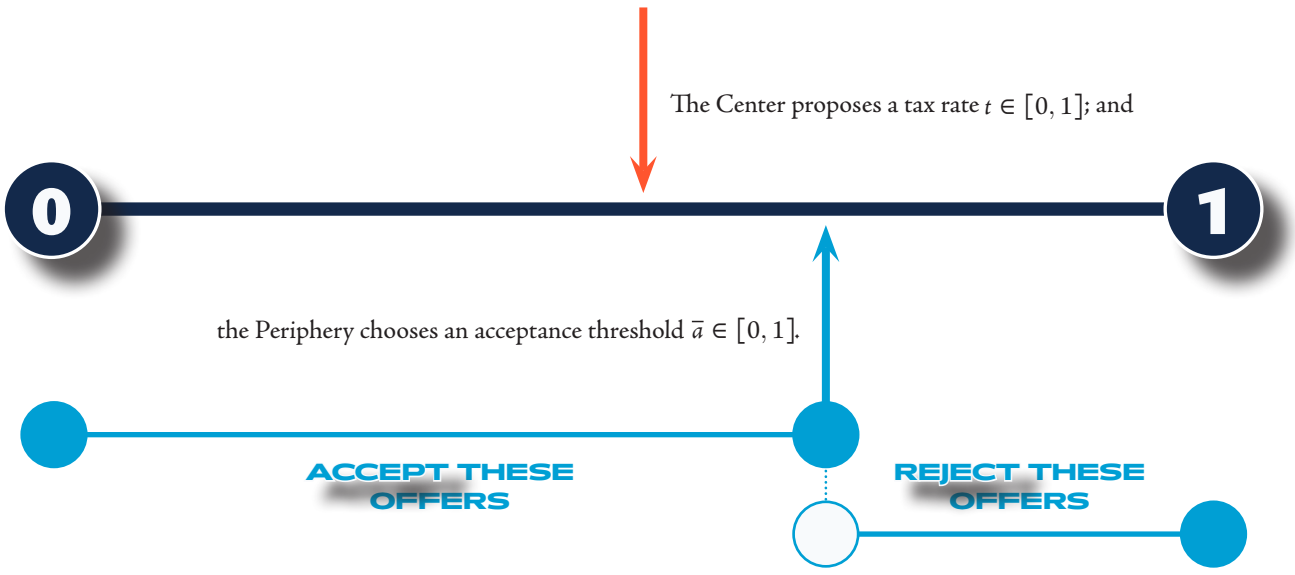
At present, Phœbustani taxes are relatively high, and one set of debates at present squares on the federal tax rate. Naturally, a sizable chunk of federal taxes wind up being sent to Daphne; citizens there favor higher federal taxes, whereas citizens in the other six cities prefer lower federal taxes. Naturally, it would be a big problem if negotiations about tax rates broke down, as that would make for a government shutdown or a vast reduction in the provision of public goods.

I've been trying to think about this problem using some of the tools we've been developing. Let's envision this as a two-player game: C (for the Center, namely Daphne) and P (for the Periphery, namely the other six cities). A given tax rate will be encoded as $t \in [0, 1]$, where $t = 0$ indicates a 0% tax rate and $t = 1$ indicates a 100% tax rate. For a given tax rate, the respective parties have utilities

$$u_C(t) = t,$$
$$u_P(t) = 1 - t.$$

In case a tax rate cannot be set in a reasonable manner, the federal government will be forced to cut services or to shut down altogether. It isn't easy for me to tell just how bad this would be, so let's err on the safe side and say that such a state of affairs is worth $d_C \in [0, 1)$ happiness points for the Center and $d_P \in [0, 1)$ happiness points for the Periphery. I think it would be bad enough that we can assume $d_C + d_P < 1$.

In my head, I'm seeing it this way:



If a deal is struck—*i.e.*, if $t \leqslant \bar{a}$— then the parties have the utilities as specified above. If a deal is not struck—*i.e.*, if $t > \bar{a}$—then the parties receive their disagreeement utilities.

Let's think:

1. Does the game just described create a bargaining problem? Why or why not?
2. Following the template above, draw the situation where $t = 1/2$ and $\bar{a} = 1/4$ and $d_C = d_P = 0$. Is the deal accepted or rejected? What are the two parties' respective utilities?
3. Same as Question 2, but with $t = 1/2$ and $\bar{a} = 3/4$.
4. Present the game in "matrix" form, where this is on the understanding that there's an uncountable number of rows and columns in said "matrix."
5. In this matrix, depict which strategy profiles have $t < \bar{a}$, which have $t = \bar{a}$, and which have $t > \bar{a}$. Write out the resepctive utilities for the two players for each of these three states of affairs.
6. Does there exist a pure-strategy Nash equilibrium where $t < \bar{a}$? If so, name one (using actual numbers). If not, why not?
7. Same as Question 6, but with $t > \bar{a}$.
8. Same as Questions 6 and 7, but with $t = \bar{a}$.
9. How many pure-strategy Nash equilibria does this game have?
10. Now suppose we allow d_C and d_P to vary, rather than being held fixed at zero. (Yes, they've been held fixed at zero for Questions 2 through 9!) How does this influence the set of pure-strategy Nash equilibria? How many pure-strategy Nash equilibria are in this game in general form?

necessary for **PASS**: get 6

sufficient for one **ALMA**: get 10

sufficient for another **ALMA**: In 250–750 words, compare the following two things: (1) the number of pure-strategy Nash equilibria in this game (*i.e.*, your answer to Question 9); and (2) the probability of obtaining a pure-strategy Nash equilibrium in a random draw from the set of all strategy profiles. Do you detect a weird tension there? You should. Like, this essay should *not* be about why you *don't* have a weird tension. I'm telling you to have a weird tension, but it should be a particular one. IYKYK. *Shut up.*

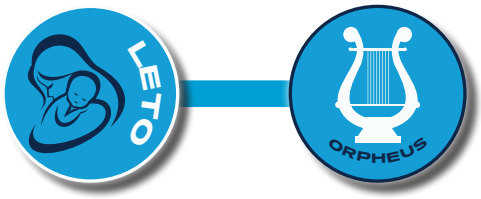
PART 2 | LAST ELECTION

It turns out that the last election in Phoebustan was a relatively straightforward regional affair: one candidate (Melia) was from Laurelia and the other (Priam) wasn't. Consequently, voters from the Sun Coast and Thessaly agreed in preferring Priam to Melia, whereas voters from Laurelia disagreed by preferring Melia to Priam.

This all seems simple, but I fear there may be a few subtleties lurking beneath the hood. So, let's study this game as follows: there are three players:

PLAYER 1 

PREFERS MELIA

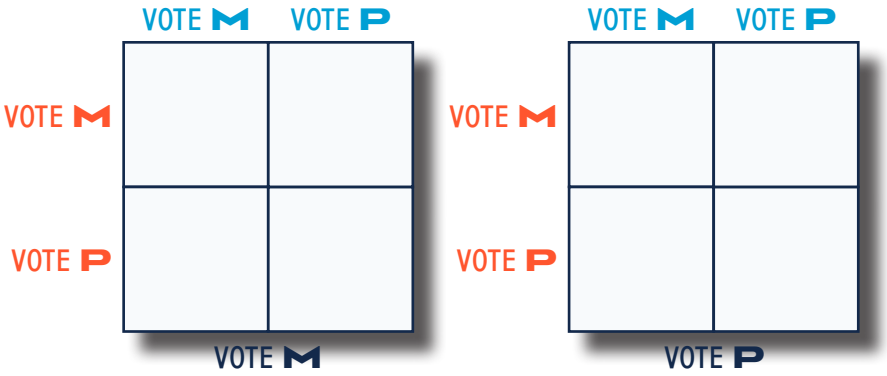
PLAYER 2 

PREFERS MELIA

PLAYER 3 

PREFERS PRIAM

Each of these three players can vote for Melia or vote for Priam. The outcome is decided by majority rule. Seeing your preferred canddiate win gets you a happiness point, whereas seeing your dis-preferred candidate win gets you zero. Here's the matrix:



Here goes:

1. For each strategy profile, determine whether Melia or Priam wins the election.
2. For each strategy profile, fill in the respective utility numbers for all three players.
3. Do any of the voter's have a strictly-dominated strategy?
4. Does your answer change if we consider *weakly*-dominated strategies? These are the same thing, but with a weak inequality in the definition rather than a strict one.
5. Identify all of the game's pure-strategy Nash equilibria.
6. We say a pure-strategy Nash equilibrium is *strong* just when there does not exist a coalition of players who could all strictly profit by deviating *together*. (In this sense, traditional Nash equilibrium focuses only on coalitions of one.) Is/are your answer(s) to Question 5 strong? If so, what coalition(s) can profitably deviate? If not, why not?

necessary for **PASS**: get 4

sufficient for one **ALMA**: get 6

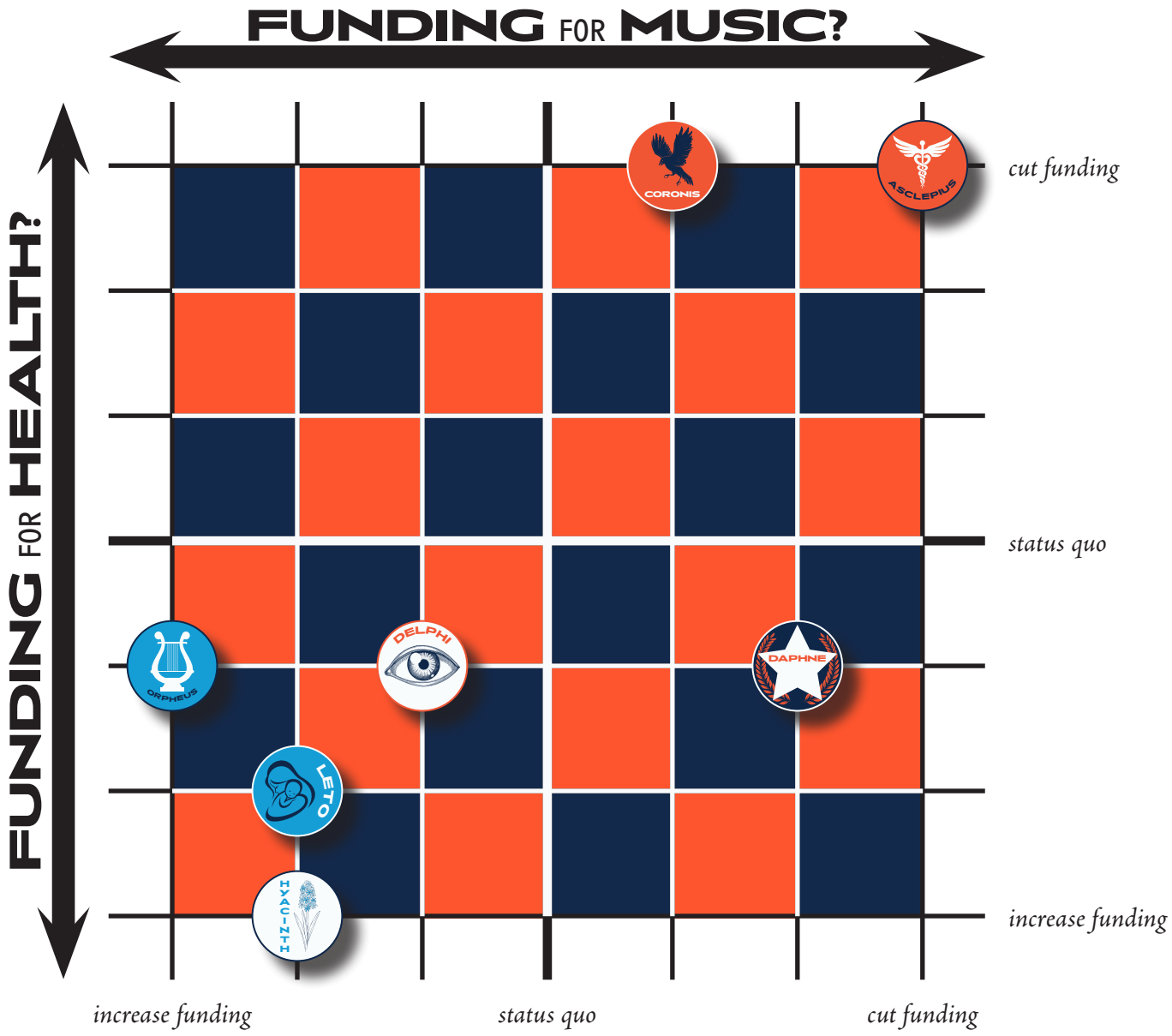
sufficient for another **ALMA**: Suppose we had the same situation but with n voters, where n is any odd counting number. Suppose that Melia is preferred by one more voter than is Priam, just as in the case for this problem. How many pure-strategy Nash equilibria does the game have? [*Make sure you finish the rest of the problem set before you try this one.*]

PART 3 | THIS ELECTION

Now this time, the incumbent Melia must stave off a fellow Letian, this one named Adonis. Since Melia and Adonis are from the same place, this next election won't be decided along regional lines. Instead, these two candidates are going to have to [*gasp*] choose a platform in a policy space.

At the moment, there seem to be two issues that differentiate Melia and Adonis: the first of these is funding music in schools and the second is funding a single-payer healthcare system with infrastructure throughout Thessaly. On the first of these, those on the left favor increasing funding for music whereas those on the right favor decreasing such funding. On the second, those on the left favor the single-payer system, whereas those on the right oppose it. The policy space looks like this:

$$X = \{-3, \ldots, 3\} \times \{-3, \ldots, 3\}$$



Suppose the cities have ideal points as depicted in the figure above, and suppose they all have spatial utilities determined by the taxicab metric:

$$u_c = -\left|x^1 - \hat{x}_c^1\right| - \left|x^2 - \hat{x}_c^2\right|.$$

Now suppose our two candidates Melia and Adonis must simultaneously choose a platform in this policy space. Let us refer to these as

$$x_M = \left(x_M^1, x_M^2\right),$$
$$x_A = \left(x_A^1, x_A^2\right).$$

Some features of the interaction:

- the cities support the candidate whose platform is closer to their ideal point;
- in case the platforms are the same distance from their ideal point, the city simply abstains;
- the winner is chosen via plurality rule, so that whichever candidate gets strictly more votes than their opponent wins;
- the candidates care only about getting elected: they receive one happiness point for winning, zero for a tie, and minus-one for losing.

Let's do it:

1. Suppose Melia chooses a platform of $x_M = (-2, -2)$ and Adonis chooses a platform of $x_A = (+2, +2)$. Which cities vote for whom, and who wins the election?
2. Same as Question 1, but let $x_M = (-2, +2)$ and $x_A = (+2, -2)$.
3. Does there exist a pure-strategy Nash equilibrium where both candidates set a platform at a city's ideal point? If so, which one(s)? If not, why not?
4. Does there exist a pure-strategy Nash equilibrium where one candidate wins and the other loses? If so, name one. If not, why not?

necessary for **PASS**: get 2

sufficient for one **ALMA**: get 4

sufficient for another **ALMA**: Look up the term "radial symmetry" and describe its relevance to this problem.