PROBLEM SET 01

THE ASSEMBLY

I am entirely too busy to get to work on the technocrats' job today! It's not every day that we have a big hairy election to send delegates to the stupid Olympian Asseembly. I only wish I could learn about the election results without having to watch some smarmy reporter with a touch screen talking about "ground games" while showing stupid diagrams of the Assembly with dots and colors and WHO CARES. Ground games! Games on the ground! And they say this as if we're supposed to think they're smart, because they're talking about ground games!

Nevertheless, it seems important to be tuned in. And even though I'd rather be at home with Laelaps playing *Δόγμα του Δολοφόνου*, watch we shall. But, it also seems that we can get our bearings as we figure out how to get these stupid technocrats off our back. Maybe we can be Those People that Practice by Not Practicing. I hate those people.

Anyway, go get yourself some popcorn, and let us see what we can do for these idiots that they could never do for themselves.

Socrates

THE MISCHIEFS OF FACTION

The idea behind the Olympian Assembly is straightforward, almost naïve: each state in the international system elects one delegate, and these thirteen delegates comprise the Assembly. Let S be the set of states in Olympia.

But one not-so-naïve development in the years of Olympian Ascendancy? The rise of three factions, each with its own unique spin on how politics in the new Assembly ought to play out!

THE BOLTS

From the grand plazas of Agoraios, the Bolts radiate their vision of strength and order. Champions of decisive leadership, they strike with the clarity and force of the thunderbolt they revere. The Bolts see the Assembly as a battleground for bold decisions and unyielding governance, aiming to secure their place at the helm of Olympia's destiny.

THE GHOSTS

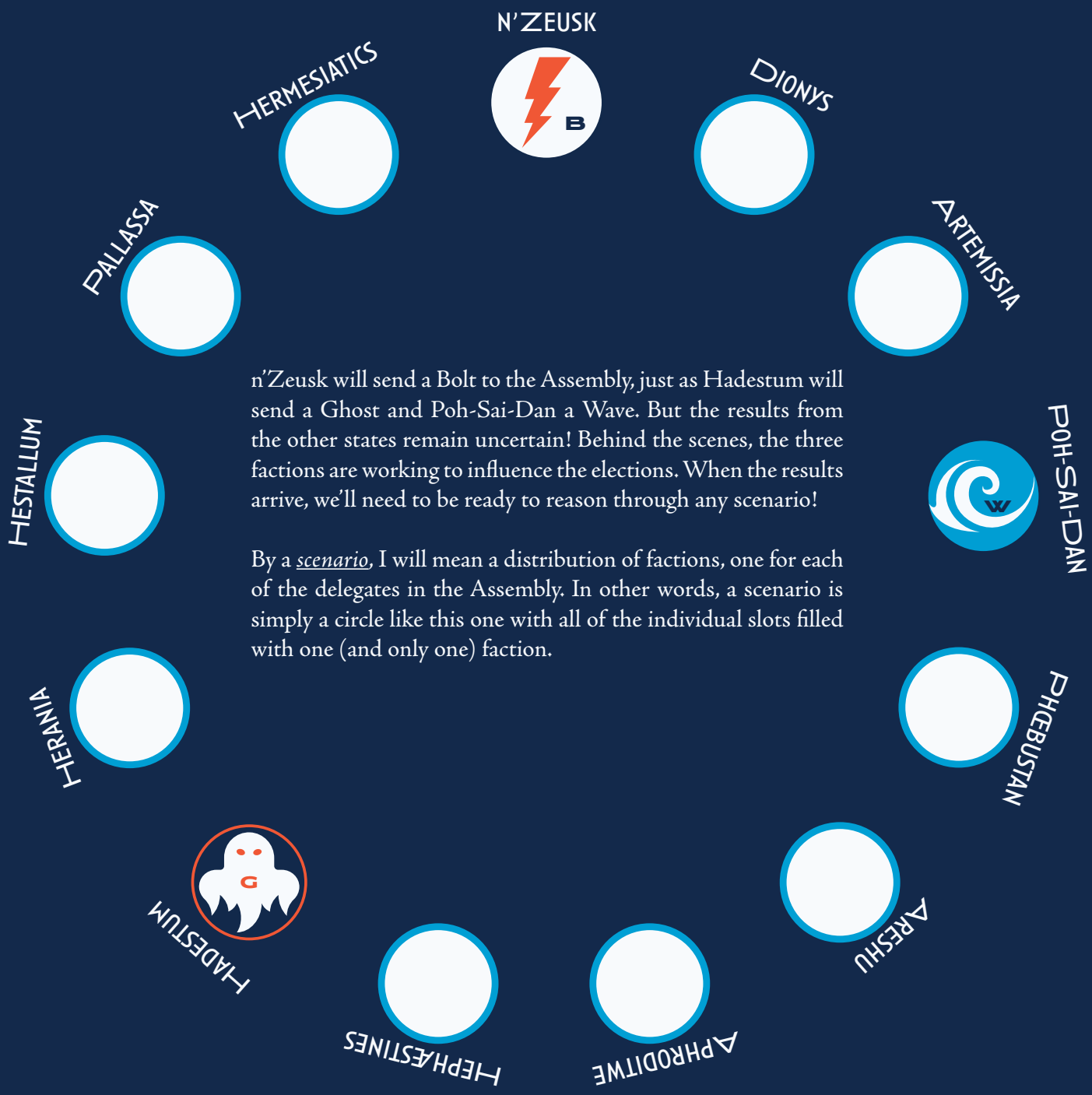
Emerging from the shadowed halls of Tartarus, the Ghosts work in whispers and shadows. Masters of strategy and subtlety, they navigate the Assembly with quiet precision, often influencing events from behind the scenes. The Ghosts play the long game, valuing endurance and strategic positioning over fleeting victories, all under the watchful orange gaze of their spectral symbol.

THE WAVES

Flowing from the coastal city of Enosichthon, the Waves carry with them the rhythm of the seas. Pragmatic and cooperative, they seek to balance power and maintain harmony in the Assembly. Like the tides they emulate, the Waves adapt to shifting dynamics, building coalitions that preserve stability even in the face of storms.

Let $F = \{B, G, W\}$ be the set of factions.

SCENARIOS IN THE ASSEMBLY



n'Zeusk will send a Bolt to the Assembly, just as Hadestum will send a Ghost and Poh-Sai-Dan a Wave. But the results from the other states remain uncertain! Behind the scenes, the three factions are working to influence the elections. When the results arrive, we'll need to be ready to reason through any scenario!

By a *scenario*, I will mean a distribution of factions, one for each of the delegates in the Assembly. In other words, a scenario is simply a circle like this one with all of the individual slots filled with one (and only one) faction.

sufficient for an **ALMA**: how is a scenario a *function*? What's the domain? What's the codomain?

sufficient for another **ALMA**: how many scenarios are possible with 13 delegates and 3 factions?

PART ONE | WORLDS, TWO WAYS

For each claim, issue a simple TRUE/FALSE evaluation, along with a sentence-long description of your reasoning/observations. [You might need to translate the claim into something a bit more precise.]

1. At least one of the Ghosts and the Waves controls more delegates than the Bolts.
2. The Ghosts and the Waves each control more delegates than the Bolts.
3. If a faction controls a majority of all delegates, then in fact it controls all of the delegates.
4. Each pair of distinct factions could form a bilateral majority coalition.
5. There exists a pair of distinct factions controlling the same number of delegates.
6. All pairs of distinct factions control the same number of delegates.
7. No faction controls more delegates than the Bolts.
8. There exists a pair of factions that could control nine (or more) delegates.
9. There is no faction with strictly fewer delegates than all other factions.
10. If any two factions each control an odd number of delegates, then so too does the third faction.

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necessary for a **PASS**: get 15

sufficient for an **ALMA**: get 20

PART TWO | THE POWER OF COALITIONS

I'm not an expert in politics—oh wait, *yes I am!*—but even I know that to pass resolutions or achieve their goals, the factions must work together—or against each other. This raises an important question: *Who should work with whom?*

- A faction like the Ghosts might seek an alliance with the Bolts to counter the Waves.
- The Bolts and Waves, traditionally rivals, might decide to set aside their differences to overpower the Ghosts.
- Or perhaps one faction might find itself isolated, unable to build a coalition at all.

In the Assembly, coalitions are everything. And to understand coalitions, we need to consider all the possible ways factions might combine their forces.

I think we can think of coalitions as subsets! Think like this: we have a set of factions, which we write out as $F = \{B, G, W\}$. Well, a coalition might be...:

- we could have a trivial coalition with just one faction, like $\{B\}$;
- we could have a coalition with two factions, like $\{G, W\}$;
- we could have a grand coalition with all factions, like $\{B, G, W\}$; and
- h*ck, we could have a coalition with nobody in it, like $\varnothing$.

So it really does seem like we can study coalitions as subsets!

Ruminate with me:

1. Write out the set of all coalitions. How many are there?
2. Write out the set of all coalitions that includes the Ghosts.
3. Write out the set of all coalitions that does not include the Waves.
4. Which coalitions control a majority in Scenario 1 from Part One?
5. Which coalitions control at least ten delegates in Scenario 2 from Part One?
6. Is it TRUE or FALSE that if a coalition is in your answer to (4), then so too is any coalition of which it is a subset? (Either discuss why it must be true, or provide a counterexample.)
7. Is it TRUE or FALSE that for any pair of coalitions, one is a subset of the other? (Same.)

But I think we can do even better by studying partitions! Why? Because partitions show how factions divide the Assembly completely and exclusively. No overlaps, no leftover states—just pure, beautiful order. A masterstroke of reasoning, I say.

So let's think about this. A partition divides the set of factions $F = \{B, G, W\}$ into non-overlapping coalitions such that every faction belongs to exactly one coalition. For example:

- possible partitions include $\{\{B, G\}, \{W\}\}$, $\{\{B\}, \{G\}, \{W\}\}$, and $\{\{B, G, W\}\}$;
- but neither $\{\{B, G\}, \{G, W\}\}$ (overlap) nor $\{\{B\}, \{W\}\}$ (incomplete coverage) is!

Oh this is going to be *pretty*, I can just *smell* it:

8. Write out all partitions of the set of factions.
9. Write out all partitions of the set of factions for which one coalition has a majority of the delegates in Scenario 1 from Part One.
10. Write out all partitions of the set of factions for which one coalition has a majority of the delegates in Scenario 2 from Part One.

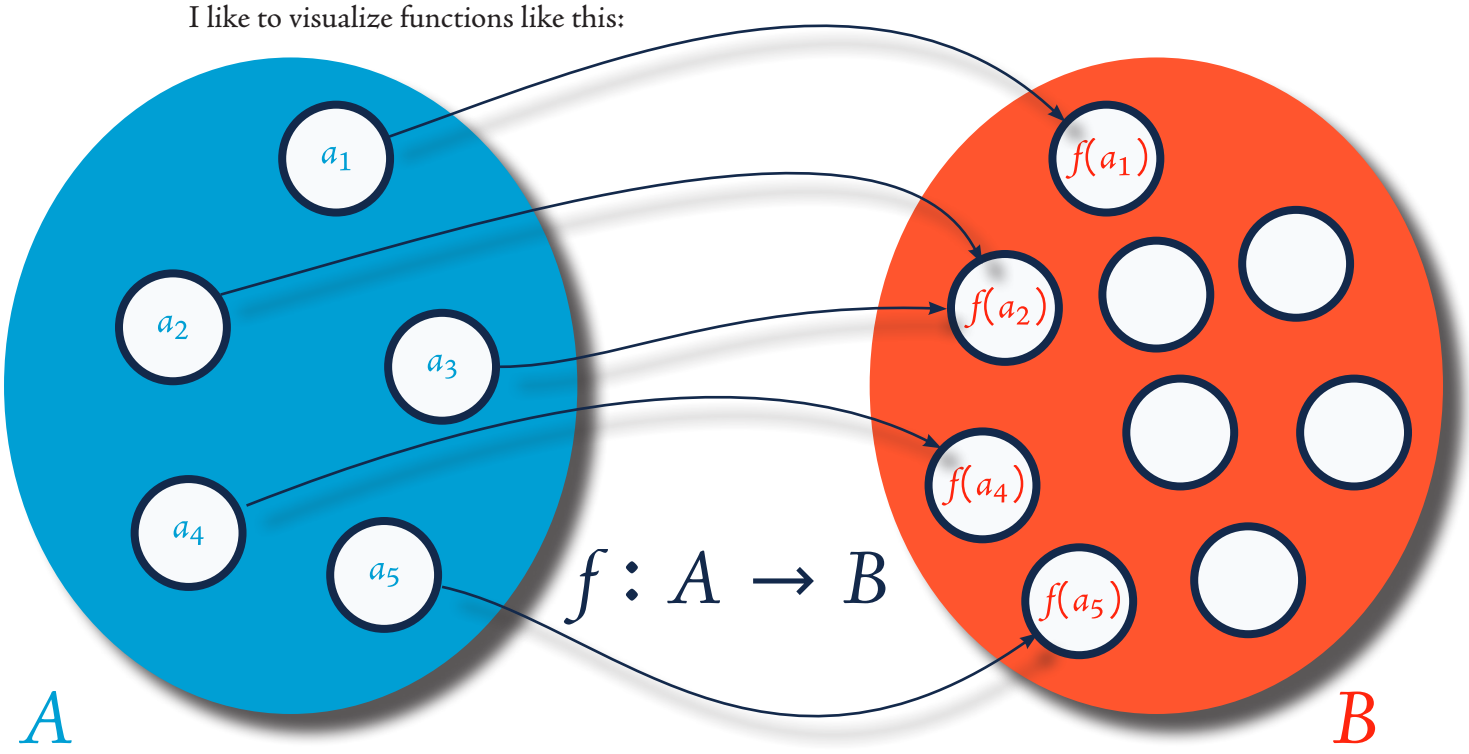
necessary for a **PASS**: get 7

sufficient for an **ALMA**: get 10

PART THREE | YOUR DESSERT COURSE

Finally, I think there's a nice way for us to summarize some of the relationships we've been uncovering in more compact terms, namely through functions! What a delight.

I like to visualize functions like this:



The unused dots on the right-side suggest there are elements of the codomain not hit by the function; this is true sometimes, and not true others! Notice also that two distinct dots in the domain might get sent to the same dot in the codomain.

Anyway, maybe we can draw these to understand things a bit better. Let's give it a try!

1. Draw the function $f : \{B, G, W\} \to \{0, 1, 2, \ldots, 13\}$ that reads in a faction and spits out the number of delegates that faction controls in Scenario 1 of Part One.
2. Suppose I didn't give you a drawing of a given scenario but instead told you that we had $f(B) + f(G) = 10$. Do you have enough information to obtain $f(W)$? If so, compute. If not, explain why.
3. Suppose I didn't give you a drawing but told you that $f(B) = 11$. Do you have enough information to obtain $f(G)$? If so, compute. If not, explain why.
4. Suppose you (again) knew $f(B) + f(G) = 10$. Do you have enough information to tell which delegates the Waves control? If so, do so. If not, explain why.
5. Draw the function $g : \{B, G, W\} \times \{B, G, W\} \to \{0, 1, \ldots, 13\}$ that reads in a pair of coalitions and spits out the number of delegates they control in Scenario 2.
6. Draw the function h : Coalitions $\to \{\mathsf{FALSE}, \mathsf{TRUE}\}$ that reads in a coalition (your answer to Question 1 of Part Two) and spits out TRUE whenever the coalition controls a majority of delegates in Scenario 1.

necessary for a **PASS**: get 3

sufficient for an **ALMA**: get 6

sufficient for another **ALMA**: are any of these injective and/or surjective?

You might feel scared right now, and that's completely normal. But trust me—*you can do this.* You might need to ask more questions than in other classes, come to office hours, use Canvas, or sit with more confusion than you're used to—but that's part of the process. Have faith in yourself and in the journey we're on together. If you're willing to invest in yourself through hard work, I promise to invest in you the same way.