# 3
# Collective Choice

Last week, we thought hard about how to model choices made by individuals. This reflects our aforementioned commitment to the tenet of *methodological individualism*. And, if we wanted, we (easily) spend the rest of the semester digging deeper into how best to model individual choices, be it theoretically or statistically. We refer to this kind of study as *decision theory*.

But, we are political scientists, and so (by definition) we care about politics of one kind or another. And, as it happens, Robinson Crusoe didn't have to think about other parties when deciding how much time to spend resting relative to the time spent harvesting oysters. The time has come to think about how to embed multiple decision-makers into some kind of social motif. Typically, we might think that this is a job of studying an intermediate amount of agency over outcomes. If an individual has precisely zero agency over her outcome, then probably won't learn much about what she will choose to do. Likewise, if an individual has full agency over her outcome, then we are in the land of decision theory again.

The space between is occupied by the theory of games and the theory of social choice. We will spend much time on game theory later, but today we will sojourn into social choice theory. Though we will not explicitly revisit social choice again after this week, it will leave a residue on how we think.[1] In particular, for those of us that do not model individuals (instead modeling states, or parties, or ethnic groups, or firms, or whatever), the question remains open about whether we can treat the entity in question as rational; indeed, we will see that *no* rational preference relation can satisfy minimal normative axioms. This is Arrow's Theorem, named after the great Kenneth Arrow, and it introduces massive doubt into the study of aggregated entities. Were it up to me, which it isn't, every first-year graduate student in the social sciences would have to read Arrow's

[1] Or at least, it *ought* to.

*Social Choice and Individual Values* [Arrow, 1951] in its entirety.

What is remarkable about most versions of the proof—one of which we will consider below—is that they reveal that choice rules violate the agency point mentioned above: if a rule satisfies minimal normative criteria, then it cannot preclude the possibility of a single actor having perfect agency over the outcome.

## *Rudiments of Social Choice*

We will now operate on a set of individuals: $N = \{1, \ldots, n\}$, with $n \geqslant 2$ and $i \in N$ being an arbitrary individual. We also study a finite set of alternatives $X$, and we define $\mathcal{X} = \mathcal{P}(X) \setminus \{\varnothing\}$ serving as the set of all nonempty subsets of $X$. We study situations under which $|X| \geqslant 3$.

Equip each individual with a regular preference relation $\succsim_i$ on $X$,[2] and let $\mathscr{R}$ represent the set of all such relations. We subsequently have the strict preference relation $\succ_i$ defined by $x \succ_i y \Leftrightarrow x \succsim_i y$ and $y \not\succsim x$ and the indifference relation $\sim_i \Leftrightarrow x \succsim_i y$ and $y \succsim_i x$. The sets of all strict preference relations and indifference relations are $\mathscr{P}$ and $\mathscr{I}$, respectively.

[2] Recall that a preference relation is regular if it is complete and transitive: that is, for any $x \in X$, we have $x \succsim_i x$ and $x \succsim_i y$ and $y \succsim_i z$ implying $x \succsim_i z$.

A preference profile $R = (\succsim_1, \ldots, \succsim_n)$ is an $n$-tuple of preference relations; we have $R \in \mathscr{R} \times \cdots \times \mathscr{R} = \mathscr{R}^n$. We will sometimes have occasion to evaluate the preference profile only on some subset of alternatives $S \subseteq X$. We define $R|_S = (\succsim_1 |_S, \ldots, \succsim_n |_S)$.

Given a preference profile $R \in \mathscr{R}^n$ and any $x, y \in X$, we define the correspondences $P, R, I$, where each works $X^2 \times \mathscr{R} \rightrightarrows N$, and in particular

$$P(x, y; R) = \{i \in N \mid x \succ_i y\},$$
$$R(x, y; R) = \{i \in N \mid x \succsim_i y\},$$
$$I(x, y; R) = \{i \in N \mid x \sim_i y\}.$$

We now define the set of all complete binary relations on $X$ as $\mathscr{B}$.[3] This will be the co-domain of our preference aggregation rules, so the idea is that we input a preference profile and output some binary relation on $X$. The only theoretical requirement for now is that the output relation be complete: for every $x, y \in X$ and every $R \in \mathscr{R}$, we need either $xf(R)y$, $yf(R)x$, or both. Thus,

[3] Just to be explicit, this means $\mathscr{R} \subset \mathscr{B}$.

**3.1 Definition** *A* preference aggregation rule *is a function,* $f : \mathscr{R}^n \to \mathscr{B}$.

The assumption on the domain is sometimes called *universal domain.*

For convenience, define the social preference relation $\mathbf{R}$, where $x\mathbf{R}y$ stands in for $xf(R)y$. The same goes for $x\mathbf{P}y$ standing in for $xf(R)y$ but not $yf(R)x$ and $x\mathbf{I}y$ standing in for $xf(R)y$ and $yf(R)x$.

Here are some examples:

1. Simple majority rule: for all $x, y \in X$, $x\mathbf{P}y$ if and only if $|P(x, y; R)| > \frac{n}{2}$.
2. Quota rules: for all $x, y \in X$, $x\mathbf{P}y$ if and only if $|P(x, y; R)| > \alpha$, where $\alpha \in [0, 1]$.
3. Unanimity: for all $x, y \in X$, $x\mathbf{P}y$ if and only if $|P(x, y; R)| = N$.
4. Pareto extension: for all $x, y \in X$, $x\mathbf{P}y$ if and only if $R(x, y; R) = N$ and $P(x, y; R) \neq \varnothing$.
5. Borda counts: for all $x, y \in X$, $x\mathbf{P}y$ if and only if $\sum_i r_i(x) < \sum_i r_i(y)$, where $r_i(x)$ is the ordinal rank of $x$ in $i$'s preference ordering on $X$.[4]

[4] This definition involves an assumption, out of convience, that the preference ordering is strict.

## *The Axiomatic Approach*

You can imagine that there are a great many preference aggregation rules. It would be hard to study them all. The pre-Arrow literature took a very one-at-a-time approach: some rule would be shown to fail some criterion of interest, and another rule would be proposed in its stead. But then, later on, this replacement rule would be shown deficient on some other metric.

This is precisely when it is helpful to think axiomatically. There are too many rules to study them all one-by-one or in this chain of slow improvement. In such circumstances, it is better to begin from first principles, stating several minimal assumptions that we think a preference aggregation rule ought to satisfy. The most foundational question, then, is whether it is possible to satisfy them all at once.

We will now introduce some axioms.

**3.2 Definition** *A preference aggregation rule $f$ satisfies:*

1. No Dictators *if there does not exist an $i \in N$ such that, for every $R \in \mathscr{R}^n$ and for any $x, y \in X$, $x \succ_i y$ implies $x\mathbf{P}y$.*
2. Unanimity *if, for every $R \in \mathscr{R}^n$ and for any $x, y \in X$, $x \succ_i y$ for all $i \in N$ implies $x\mathbf{P}y$.*
3. Independence of Irrelevant Alternatives (IIA) *if, for every $R, R' \in \mathscr{R}$ and for any $x, y \in X$, $R|_{\{x,y\}} = R'|_{\{x,y\}}$ implies that $f(R)|_{\{x,y\}} = f(R')|_{\{x,y\}}$.*

4. Transitivity *if, for every* $x, y, z \in X$ *and every* $R \in \mathscr{R}^n$, $xf(R)y$ *and* $yf(R)z$ *implies* $xf(R)z$.

We could just as easily have focused on linear orders—*i.e.*, made $\mathscr{R}$ the codomain of $f$, rather than its superset $\mathscr{B}$—and skipped a definition of transitivity. But, this approach will be more flexible for the rest of the lecture.

These are relatively minimal. The No Dictators requirement ensures that no one individual is guaranteed to exert complete agency over the outcome. (You may also interpret this as: wow, that is a really extreme definition of what a dictator is! Thus, to exclude such an extreme kind of dictator is quite mild.) The Unanimity requirement means that means that $x$ must be strictly socially preferred to $y$ if all individuals strictly prefer $x$ to $y$. (Again, it would be really bad if society couldn't respond to such insanely strong agreement among the citizens.) The IIA requirement means that the aggregated preferences between $x$ and $y$ depend *only* on the binary rankings between $x$ and $y$. And, given that all $f$s are defined on the set of complete binary relations, Transitivity is our measure of rationality—all of basic order theory depends on transitivity, as we saw last week.

Yet, it is hard to find a rule that satisfies all four. For example, let us try $N = \{1, 2, 3\}$, $X = \{w, x, y, z\}$, and $f$ as the Borda count, with the following preference profile:

$$w \succ_1 x \succ_1 y \succ_1 z,$$
$$y \succ_2 z \succ_2 x \succ_2 w,$$
$$z \succ_3 y \succ_3 w \succ_3 x.$$

You can see for yourself that this means $w\mathbf{P}x$. Now consider a second profile:

$$y \succ'_1 z \succ'_1 w \succ'_1 x,$$
$$x \succ'_2 y \succ'_2 z \succ'_2 w,$$
$$z \succ'_3 y \succ'_3 w \succ'_3 x.$$

Now $x\mathbf{P}'w$. But this violates IIA: observe that each individual ranking between $w$ and $x$ remained unchanged in the two profiles: 1 prefers $w$ to $x$; 2 prefers $x$ to $w$, and 3 prefers $w$ to $x$.

This example helps to highlight that IIA is more subtle than No Dictators and Unanimity. Essentially, IIA asks that we leave out any *cardinal information* in the preferences, preferring only to rest on *ordinal information*. Thus, if Tallahassee is my absolute favorite place in the world and Gainesville is my absolute least favorite place, then this intensity matters no more than if they were my least two favorite places, nor my favorite two places, nor just two places in the middle. But, as you might imagine, IIA is there for technical reasons more than normative reasons.

Anyway, we have accumulated sufficient notation to state one of the most important theorems in social science and one of the most important thoughts in normative democractic theory:

**3.3 Theorem (Arrow)** *No preference aggregation rule simultaneously satisfies No Dictators, Unanimity, IIA, and Transitivity.*

Well, that stinks.

Before proving the theorem, let's try to visualize what has to happen for all good things to go together. To do so, we will consider a simple example: $N = 2$ and $X = \{x, y, z\}$. We will in particular focus on strict preference orderings, so that for every $a, b \in X$, either $a \succ_i b$ or $b \succ_i a$ (but not both!). With two individuals and three alternatives, there are 36 possible combinations. We will arrange these in

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | | | | | | |
| $xzy$ | | | | | | |
| $yxz$ | $f(R_{31})$ | $f(R_{32})$ | $f(R_{33})$ | $f(R_{34})$ | $f(R_{35})$ | $f(R_{36})$ |
| $yzx$ | | | | | | |
| $zxy$ | | | | | | |
| $zyx$ | | | | | | |

Table 3.1: The structure of the problem.

tabular format in Table 3.1. Again, there are 36 cells in play. The rows denote $\succ_1$ and the columns denote $\succ_2$. To show you how the table works, I've filled in what should go into each cell: a binary relation as a function of the two profiles. Here we are holding $y \succ_1 x \succ_1 z$ fixed and evaluating across possible $\succ_2$ relations. The subscripts refer to rows, then columns.

We will now consider how each of the conditions works in this graphical format. Most fundamentally, our assumption of universal domain means that every cell must be filled in: there given any two rankings, society must be able to produce a social preference. Transitivity comes next. It means that whatever relation we wind up putting into each of the cells must be transitive. We will worry about that once we've done some filling in!

It will be easiest to begin from Unanimity. Let us consider all of the admissable rankings given our small example; see Table 3.2. Here we use the Unanimity conditions to demonstrate what conditions *must* hold. Sometimes, Unanimity offers no bite: for exmaple, if $\succ_1$ is $yzx$ and $\succ_2$ is $xzy$, then the two voters don't agree on anything, so nothing is required of the preference aggregation rule. On the other hand, along the diagonal of this matrix, the two individuals agree perfectly, and the Unanimity axiom requires that the preference

Table 3.2: Building from the Unanimity requirement first.

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | $xyz$ | $xy$, $xz$ | $xz$, $yz$ | $yz$ | $xy$ | |
| $xzy$ | $xy$, $xz$ | $xzy$ | $xz$ | | $xy$, $zy$ | $zy$ |
| $yxz$ | $xz$, $yz$ | $xz$ | $yxz$ | $yz$, $yx$ | | $yx$ |
| $yzx$ | $yz$ | | $yx$, $yz$ | $yxz$ | $zx$ | $yx$, $yx$ |
| $zxy$ | $xy$ | $xy$, $zy$ | | $zx$ | $zxy$ | $zx$, $zy$ |
| $zyx$ | | $zy$ | $yx$ | $yx$, $zx$ | $zx$, $zy$ | $zyx$ |

aggregation take on the same form. Sometimes we learn only one binary preference, and sometimes we learn two. Now, in the cases where we learn two separate ones (that is, the cells with commas in them), you'll note that Transitivity requirements never kick in: all of the comma cells are of the form ($xy$, $xz$)—so that $x$ must always be preferred—or ($xz$, $yz$)—so that $z$ must never be preferred. Neither of these forms are of the ($xy$, $yz$) form that lets Transitivity bite. In the diagonial unanimity cases, Transitivity is respected, but it was Unanimity that created the complete relation, not Transitivity.

Very well. We can now use some of what we've learned in tandem with IIA. If you go back and check the definition, you'll see that IIA is a pair-by-pair requirement, so we will have to use it three times: once between $x$ and $y$; another between $x$ and $z$; and a third between $y$ and $z$. Let's do $x$ and $y$ first. What IIA tells us is that every time we learn $xy$ from one set of preferences, we learn it in every other set of preferences with the same rankings between $x$ and $y$. For example, consider the first row, fourth column, which contains the requirement $yz$. Let us add an assumed value here, just to see how it works. In particular, suppose that in the first row and fourth column, we add $x\mathbf{P}y$, so that the cell is now $xyz$.[5] This means that *every* time we have $x \succ_1 y$ and $y \succ_2 x$ (as is the case in this cell), we again have $x\mathbf{P}y$.

[5] We were going to have to add something about $x$ and $y$ here, anyway, per universal domain.

Table 3.3: Adding IIA on the assumption that $xf(xyz, yzx)y$.

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | $xyz$ | $xy$, $xz$ | $xyz$ | $xyz$ | $xy$ | $xy$ |
| $xzy$ | $xy$, $xz$ | $xzy$ | $xz$, $xy$ | $xy$ | $xy$, $zy$ | $zy$, $xy$ |
| $yxz$ | $xz$, $yz$ | $xz$ | $yxz$ | $yz$, $yx$ | | $yx$ |
| $yzx$ | $yz$ | | $yx$, $yz$ | $yxz$ | $zx$ | $yx$, $yx$ |
| $zxy$ | $xy$ | $xy$, $zy$ | $xy$ | $zxy$ | $zxy$ | $zxy$ |
| $zyx$ | | $zy$ | $yx$ | $yx$, $zx$ | $zx$, $zy$ | $zyx$ |

Then so too must we add $x\mathbf{P}y$ in cells $(1, 3)$ (forming a linked $xyz$), $(1, 6)$, $(2, 3)$, $(2, 4)$ $(2, 6)$, $(5, 3)$, $(5, 4)$ (making the chain $zxy$), and $(5, 6)$ (forming a linked $zxy$).

Now, we used Transitivity in cell $(1, 4)$ to go from $yz$ (by Unanimity) and $xy$ (by hypothesis) to $xz$. Well, now we have another inference to make from IIA: every time $\succ_1$ features $x \succ_1 z$ and $\succ_2$ features $z \succ_2 x$, we need $x\mathbf{P}z$. This happens in cells $(1,5)$, $(1,6)$, $(2,4)$, $(2,5)$

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | $xyz$ | $xy$, $xz$ | $xyz$ | $xyz$ | $xy$, $xz$ | $xy$, $xz$ |
| $xzy$ | $xy$, $xz$ | $xzy$ | $xz$, $xy$ | $xy$, $xz$ | $xzy$ | $xzy$ |
| $yxz$ | $xz$, $yz$ | $xz$ | $yxz$ | $yxz$ | $xz$ | $yxz$ |
| $yzx$ | $yz$ | | $yx$, $yz$ | $yzx$ | $zx$ | $yx$, $yx$ |
| $zxy$ | $xy$ | $xy$, $zy$ | $xy$ | $zxy$ | $zxy$ | $zxy$ |
| $zyx$ | | $zy$ | $yx$ | $yx$, $zx$ | $zx$, $zy$ | $zyx$ |

Table 3.4: Using IIA from our $x\mathbf{P}z$ deduction.

(making the chain $xzy$), $(2,6)$ (making the chain $xzy$), $(3,4)$ (making the chain $xyz$), $(3,5)$, and $(3,6)$ (making the chain $yxz$).

But, you'll note also that we used transitivity in cell $(5,4)$ to get $z\mathbf{P}y$. This means that we must again have $z\mathbf{P}y$ each time $z \succ_1 y$ and $y \succ_2 z$. (Are you beginning to see what's going on here?) This applies

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | $xyz$ | $xy$, $xz$ | $xyz$ | $xyz$ | $xy$, $xz$ | $xy$, $xz$ |
| $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ |
| $yxz$ | $xz$, $yz$ | $xz$ | $yxz$ | $yxz$ | $xz$ | $yxz$ |
| $yzx$ | $yz$ | | $yx$, $yz$ | $yzx$ | $zx$ | $yx$, $yx$ |
| $zxy$ | $xy$, $zy$ | $xy$, $zy$ | $xy$, $zy$ | $zxy$ | $zxy$ | $zxy$ |
| $zyx$ | $zy$ | $zy$ | $zyx$ | $zyx$ | $zx$, $zy$ | $zyx$ |

Table 3.5: Using IIA from our $z\mathbf{P}y$ deduction.

to cells $(2,1)$, $(2,3)$, $(2,4)$, $(5,1)$, $(5,3)$, $(6,1)$, $(6,3)$, and $(6,4)$. You'll note that, by now, we've got one entire row that follows $\succ_1$ perfectly (the second row) and one entire column that does so, too (the fourth column).

We saw in cell $(3,6)$ that $y\mathbf{P}z$ if $y \succ_1 z$ and $z \succ_2 y$. We also saw in cell $(6,3)$ that $z\mathbf{P}x$ if $z \succ_1 x$ and $x \succ_2 z$. Let's fill those in, which is executed in Table 3.6 for those of you keeping score. By now the

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | $xyz$ | $xyz$ | $xyz$ | $xyz$ | $xyz$ | $xyz$ |
| $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ |
| $yxz$ | $xz$, $yz$ | $xz$, $yz$ | $yxz$ | $yxz$ | $xz$, $yz$ | $yxz$ |
| $yzx$ | $yzx$ | $yzx$ | $yzx$ | $yzx$ | $yzx$ | $yzx$ |
| $zxy$ | $zxy$ | $zxy$ | $zxy$ | $zxy$ | $zxy$ | $zxy$ |
| $zyx$ | $zy$, $zx$ | $zy$, $zx$ | $zyx$ | $zyx$ | $zx$, $zy$ | $zyx$ |

Table 3.6: Using IIA from our $y\mathbf{P}z$ and $z\mathbf{P}x$ deductions.

original pattern we saw is beginning to take an even stronger shape.

Finally, in cell $(4, 5)$ we saw $y\mathbf{P}x$ when $y \succ_1 x$ and $x \succ_2 y$. This lets us fill in the final little bits in the table, resulting in What has hap-

Table 3.7: Individual 1 is a dictator.

| | $xyz$ | $xzy$ | $yxz$ | $yzx$ | $zxy$ | $zyx$ |
|---|---|---|---|---|---|---|
| $xyz$ | $xyz$ | $xyz$ | $xyz$ | $xyz$ | $xyz$ | $xyz$ |
| $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ | $xzy$ |
| $yxz$ | $yxz$ | $yxz$ | $yxz$ | $yxz$ | $yxz$ | $yxz$ |
| $yzx$ | $yzx$ | $yzx$ | $yzx$ | $yzx$ | $yzx$ | $yzx$ |
| $zxy$ | $zxy$ | $zxy$ | $zxy$ | $zxy$ | $zxy$ | $zxy$ |
| $zyx$ | $zyx$ | $zyx$ | $zyx$ | $zyx$ | $zyx$ | $zyx$ |

pened, then, was that our innocuous assumption in cell $(1, 4)$ really got out of hand. From it, IIA and Transitivity worked together several times to lead us to conclusions. Those conclusions, in turn, worked with IIA and Transitivity to yield more conclusions. And so on, all the way until we realized that in letting 1 beat 2 in a single instance of conflict, we implicitly let 1 beat 2 in *all* instances of conflict.[6]

[6] The same thing would have happened in reverse if we had chosen 2 instead of 1 at the outset.

We have shown the result with $n = 2$ and $|X| = 3$. But how will we make it work more generally? It turns out that we will work with the same logic as above, but in a much more general way.

## *Proof of Arrow's Theorem*

**3.3 Theorem (Arrow)** *No preference aggregation rule simultaneously satisfies No Dictators, Unanimity, IIA, and Transitivity.*

This version of the proof is due to Fey [2014]. It follows in seven steps.[7] The general goal is to demonstrate that if a preference aggregation rule satisfies Unanimity, IIA, and Transitivity, then it cannot satisfy No Dictators. To fail No Dictators, there must be some individual, call her $i^*$, whose strictly preferences are always strictly reflected in the preference aggregation rule—that is, there must be some individual for whom $a \succ_{i^*} b \Rightarrow a\mathbf{P}b$.

[7] There are many ways to prove Arrow's theorem. Some involve many steps. Others involve a single step. This one is great from a pedagogical standpoint and imposes fewer assumptions, but at the cost of more steps.

A quick definition: we say a voter $i$ is decisive for some $x$ over some $y \neq x$ if $x \succ_i y$ implies $x\mathbf{P}y$. If voter $i$ is decisive for every $x$ over every $y \neq x$, then she is a dictator—*i.e,* she is the violator of No Dictators.

### *Step 1: identify i**

Let $a, b \in X$ be two arbitrary alternatives. Consider a profile where all voters rank $a$ ahead of $b$. I depict this at right; the vertical dots capture fixed but arbitrary rankings over all the other alternatives. By

IIA, the only thing that matters is the fact that $a$ is ahead of $b$ in each individual ranking.

Just to highlight that IIA helps us here, let's *elevate* alternatives $a$ and $b$ to the top of the rankings. That is, we'll consider a particular instantiation of $a$ being unanimously preferred to $b$: the one where all voters rank $a$ first and $b$ second. By IIA, whatever choice is made from the elevated profile will be the same as the choice from an arbitrary profile: that is, these two tables must yield the same ranking over $a$ and $b$.

| $\succsim_1$ | $\cdots$ | $\succsim_n$ |
|---|---|---|
| $\vdots$ | | $\vdots$ |
| $a$ | $\cdots$ | $a$ |
| $\vdots$ | | $\vdots$ |
| $b$ | $\cdots$ | $b$ |
| $\vdots$ | | $\vdots$ |

Table 3.8: Step 1: the arbitrary profile

| $\succsim_1$ | $\cdots$ | $\succsim_n$ |
|---|---|---|
| $a$ | $\cdots$ | $a$ |
| $b$ | $\cdots$ | $b$ |
| $\vdots$ | | $\vdots$ |

Table 3.9: Step 1: the profile after the *ab*-elevation.

| $\succsim_1$ | $\cdots$ | $\succsim_n$ |
|---|---|---|
| $b$ | $\cdots$ | $b$ |
| $a$ | $\cdots$ | $a$ |
| $\vdots$ | | $\vdots$ |

Table 3.10: Step 1: the "opposite unanimous" profile.

We will focus on profiles of this type on the understanding that we can send $a$ or $b$ (and sometimes $c$) "down" the profile while maintaining the ordering.

In particular, for the profile depicted in Table 3.9, Unanimity tells us that we must have $a\mathbf{P}b$, because every voter ranks $a$ ahead of $b$. Likewise, Unanimity states that the profile in Table 3.10 must yield $b\mathbf{P}a$, because every voter ranks $b$ ahead of $a$.

Now, consider the following procedure:

1. Begin from the profile in Table 3.9, which must yield $a\mathbf{P}b$ by Unanimity.

2. Holding all other aspects of the profile fixed, switch the $ab$ ordering for Voter 1. We now have a profile where Voter 1 ranks $b$ first, $a$ second, and all other alternatives in fixed and arbitrary order, whereas Voters 2 through $n$ rank $a$ first, $b$ second, and all other alternatives in fixed and arbitrary order. Call this $R'_1$.

3. Now,

   (a) if $f(R'_1)$ yields $b\mathbf{R}a$, then identify Voter 1 as $i^*$ and proceed to step 2.

   (b) if $f(R'_1)$ still yields $a\mathbf{P}b$, switch Voter 2's ranking of $a$ and $b$ (so that now $b \succ_1 a$ AND $b \succ_2 a$) and call the resulting profile $R'_2$.

We repeat Step (3)—each time adding a new voter strictly preferring $b$ to $a$—until we find the voter that "switches" the social preference; because the eventually stopping point of this procedure is the profile in Table 3.10 (which yields $b\mathbf{R}a$), we know that such a switch ultimately occurs. That is, Voter $i^*$ is the voter for whom the profile in Table 3.11 yields $a\mathbf{P}b$, but the profile in Table 3.12 yields $b\mathbf{R}a$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i*-1}$ | $\succsim_{i*}$ | $\succsim_{i*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $b$ | $\cdots$ | $b$ | $a$ | $a$ | $\cdots$ | $a$ |
| $a$ | $\cdots$ | $a$ | $b$ | $b$ | $\cdots$ | $b$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.11: Profile $R'_{i*-1}$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i*-1}$ | $\succsim_{i*}$ | $\succsim_{i*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $b$ | $\cdots$ | $b$ | $b$ | $a$ | $\cdots$ | $a$ |
| $a$ | $\cdots$ | $a$ | $a$ | $b$ | $\cdots$ | $b$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.12: Profile $R'_{i*}$.

*Step 2: show Voter $i^*$ is decisive for $b$ over any $c \neq a, b$*

Consider the profile in Table 3.13—in particular, notice that it has the same $ab$ rankings as does the profile in Table 3.11. By IIA, we must have $a\mathbf{P}b$ in this profile. Notice also that this profile features $b \succ_i c$ for all $i$, so Unanimity tells us that we must have $b\mathbf{P}c$, too. Linking $a\mathbf{P}b$ and $b\mathbf{P}c$ and invoking Transitivity, we infer that $a\mathbf{P}c$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i^*-1}$ | $\succsim_{i^*}$ | $\succsim_{i^*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $b$ | $\cdots$ | $b$ | $a$ | $a$ | $\cdots$ | $a$ |
| $c$ | $\cdots$ | $c$ | $b$ | $b$ | $\cdots$ | $b$ |
| $a$ | $\cdots$ | $a$ | $c$ | $c$ | $\cdots$ | $c$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.13: The first profile for Step 2.

Now consider the profile in Table 3.14. Here we use $b/c$ notation to mean that, in the spot in question, alternatives $b$ and $c$ can be ranked arbitrarily in the indicated spot—that is, $b$ can be ranked ahead of $c$, or $c$ can be ranked ahead of $b$, or the voter can be indifferent between the two. The thing that matters here is that $b/c$ are still ranked ahead of $a$ for the first $i^* - 1$ voters and that $b/c$ are ranked after $a$ for voters $i^* + 1$ through $n$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i^*-1}$ | $\succsim_{i^*}$ | $\succsim_{i^*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $b/c$ | $\cdots$ | $b/c$ | $b$ | $a$ | $\cdots$ | $a$ |
| $a$ | $\cdots$ | $a$ | $a$ | $b/c$ | $\cdots$ | $b/c$ |
| $\vdots$ | | $\vdots$ | $c$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.14: The second profile for Step 2.

In the set of profiles captured in Table 3.14, the $ba$ rankings are the same as in Table 3.12. By IIA, we must have $b\mathbf{R}a$. And, the $ac$ rankings are the same as they are in Table 3.13, so we must also have $a\mathbf{P}c$. Linking $b\mathbf{R}a$ and $a\mathbf{P}c$ and invoking Transitivity,[8] we conclude that $b\mathbf{P}c$. Since the $bc$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $b \succ_{i^*} c$ falls into the class of profiles shown in Table 3.14 as far as $b$ and $c$ are concerned. Therefore, IIA demands that any time $b \succ_{i^*} c$, we have $b\mathbf{P}c$. We conclude that Voter $i^*$ is decisive for $b$ over any $c \notin \{a, b\}$.

[8] Recall from last problem set that, if $\succsim$ is transitive, then $x \succsim y$ and $y \succ z$ implies $x \succ z$.

*Step 3: show Voter $i^*$ is decisive for $a$ over any $c \neq a, b$*

Now consider the set of profiles in Table 3.15. Observe that in this set of profiles, $a \succ_i b$ for all $i$, so Unanimity tells us $a\mathbf{P}b$. Likewise, because Voter $i^*$ is decisive for $b$ over $c$ (per Step 2) and $b \succ_{i^*} c$ in this set of profiles, we have $b\mathbf{P}c$. Linking $a\mathbf{P}b$ and $b\mathbf{P}c$ and invoking Transitivity, we infer $a\mathbf{P}c$. Since the $ac$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $a \succ_{i^*} c$ falls into the class of profiles shown in Table 3.15 as far as $a$ and $c$ are concerned. Therefore, IIA demands that any time $a \succ_{i^*} c$, we have $a\mathbf{P}c$. We conclude that Voter $i^*$ is decisive for $a$ over any $c \notin \{a, b\}$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i^*-1}$ | $\succsim_{i^*}$ | $\succsim_{i^*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $a/c$ | $\cdots$ | $a/c$ | $a$ | $a/c$ | $\cdots$ | $a/c$ |
| $b$ | $\cdots$ | $b$ | $b$ | $b$ | $\cdots$ | $b$ |
| $\vdots$ | | $\vdots$ | $c$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.15: The profile for Step 3.

*Step 4: show Voter $i^*$ is decisive for any $c \neq a, b$ over $a$*

Consider the profile in Table 3.16. Observe that it ranks $a$ and $b$ the same as the profile in Table 3.11; IIA tells us we must have $a\mathbf{P}b$. And, it has $c \succ_i a$ for all $i$, so Unanimity tells us we must have $c\mathbf{P}a$. Linking $c\mathbf{P}a$ and $a\mathbf{P}b$ and invoking Transitivity, we infer $c\mathbf{P}b$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i^*-1}$ | $\succsim_{i^*}$ | $\succsim_{i^*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $b$ | $\cdots$ | $b$ | $c$ | $c$ | $\cdots$ | $c$ |
| $c$ | $\cdots$ | $c$ | $a$ | $a$ | $\cdots$ | $a$ |
| $a$ | | $a$ | $b$ | $b$ | | $b$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.16: The first profile for Step 4.

Now consider the class of profiles depicted in Table 3.17. These profiles all rank $a$ and $b$ the same as the profile in Table 3.12, so IIA tells us that we must have $b\mathbf{R}a$. And, they all rank $b$ and $c$ the same as the profile in Table 3.16, so IIA tells us that we must have $c\mathbf{P}b$. Linking $c\mathbf{P}b$ and $b\mathbf{R}a$ and invoking Transitivity, we infer $c\mathbf{P}a$. Since the $ac$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $c \succ_{i*} a$ falls into the class of profiles shown in Table 3.17 as far as $a$ and $c$ are concerned. Therefore, IIA demands that any time $c \succ_{i*} a$, we have $c\mathbf{P}a$. We conclude that Voter $i^*$ is decisive for $c \notin \{a, b\}$ over $a$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i*-1}$ | $\succsim_{i*}$ | $\succsim_{i*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $b$ | $\cdots$ | $b$ | $c$ | $a/c$ | $\cdots$ | $a/c$ |
| $a/c$ | $\cdots$ | $a/c$ | $b$ | $b$ | $\cdots$ | $b$ |
| $\vdots$ | | $\vdots$ | $a$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.17: The second profile for Step 4.

*Step 5: show Voter i* is decisive for any $c \neq a, b$ over b*

Consider the profile in Table 3.18. Since Voter $i^*$ is decisve for $c$ over $a$, we have $c\mathbf{P}a$. And, observe that this profile has $a \succ_i b$ for all $i$, so Unanimity gives us $a\mathbf{P}b$. Linking $c\mathbf{P}a$ and $a\mathbf{P}b$ and invoking Transitivity, we infer $c\mathbf{P}b$. Since the $bc$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $c \succ_{i*} b$ falls into the class of profiles shown in Table 3.18 as far as $b$ and $c$ are concerned. Therefore, IIA demands that any time $b \succ_{i*} c$, we have $c\mathbf{P}b$. We conclude that Voter $i^*$ is decisive for $c \notin \{a, b\}$ over $b$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i*-1}$ | $\succsim_{i*}$ | $\succsim_{i*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $a$ | $\cdots$ | $a$ | $c$ | $a$ | $\cdots$ | $a$ |
| $b/c$ | $\cdots$ | $b/c$ | $a$ | $b/c$ | $\cdots$ | $b/c$ |
| $\vdots$ | | $\vdots$ | $b$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.18: The profile for Step 5.

*Step 6: show Voter i* is decisive for a over b and b over a*

Consider the profile in Table 3.19. Since Voter $i^*$ is decisive for $a$ over $c$ (Step 3) and decisive for $c$ over $b$ (Step 5), we have $a\mathbf{P}c$ and $c\mathbf{P}b$. Linking these and invoking Transitivity, we infer $a\mathbf{P}b$. Since the $ab$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $a \succ_{i*} b$ falls into the class of profiles shown in Table 3.19 as far as $a$ and $b$ are concerned. Therefore, IIA demands that any time $a \succ_{i*} b$, we have $a\mathbf{P}b$. We conclude that Voter $i^*$ is decisive for $a$ over $b$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i*-1}$ | $\succsim_{i*}$ | $\succsim_{i*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $a/b$ | $\cdots$ | $a/b$ | $a$ | $a/b$ | $\cdots$ | $a/b$ |
| $\vdots$ | | $\vdots$ | $c$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $b$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.19: The first profile for Step 6.

Now consider the profile in Table 3.19. Since Voter $i^*$ is decisive for $b$ over $c$ (Step 2) and decisive for $c$ over $a$ (Step 4), we have $b\mathbf{P}c$ and $c\mathbf{P}a$. Linking these and invoking Transitivity, we infer $b\mathbf{P}a$. Since the $ab$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $b \succ_{i*} a$ falls into the class of profiles shown in Table 3.19 as far as $a$ and $b$ are concerned. Therefore, IIA demands that any time $b \succ_{i*} a$, we have $b\mathbf{P}a$. We conclude that Voter $i^*$ is decisive for $b$ over $a$.

| $\succsim_1$ | $\cdots$ | $\succsim_{i*-1}$ | $\succsim_{i*}$ | $\succsim_{i*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $a/b$ | $\cdots$ | $a/b$ | $b$ | $a/b$ | $\cdots$ | $a/b$ |
| $\vdots$ | | $\vdots$ | $c$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $a$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.20: The second profile for Step 6.

### Step 7: show Voter $i^*$ is a dictator

To this point, we have shown that if $x \in \{a, b\}$ or $y \in \{a, b\}$,[9] then Voter $i^*$ is decisive for $x$ over $y$ and for $y$ over $x$. The remaining case is when $x \notin \{a, b\}$ and $y \notin \{a, b\}$.

To handle this case, consider the class of profiles in Table 3.21. By Step 4, Voter $i^*$ is decisive for $x$ over $a$, so $x\mathbf{P}a$. By Step 6, Voter $i^*$ is decisive for $a$ over $b$, so $a\mathbf{P}b$. By Step 2, Voter $i^*$ is decisive for $b$ over $y$, so $b\mathbf{P}y$. Linking these together and repeatedly invoking Transitivity, we infer $x\mathbf{P}y$. Since the $xy$ comparisons are arbitrary for all voters other than $i^*$, *any* profile with $x \succ_{i^*} y$ falls into the class of profiles shown in Table 3.21 as far as $x$ and $y$ are concerned. Therefore, IIA demands that any time $x \succ_{i^*} y$, we have $x\mathbf{P}y$. We conclude that Voter $i^*$ is decisive for any $x$ over any $y \neq x$. Hence, Voter $i^*$ is a dictator.

[9] If $x \in \{a, b\}$ but $y \notin \{a, c\}$, then $y$ is the arbitrary $c$. Likewise, if $y \in \{a, b\}$ but $x \notin \{a, b\}$, then $x$ is the arbitrary $c$. If $x \in \{a, b\}$ and $y \in \{a, b\}$, then we're in the Step 6 case.

| $\succsim_1$ | $\cdots$ | $\succsim_{i^*-1}$ | $\succsim_{i^*}$ | $\succsim_{i^*+1}$ | $\cdots$ | $\succsim_n$ |
|---|---|---|---|---|---|---|
| $x/y$ | $\cdots$ | $x/y$ | $x$ | $x/y$ | $\cdots$ | $x/y$ |
| $a/b$ | $\cdots$ | $a/b$ | $a$ | $a/b$ | $\cdots$ | $a/b$ |
| $\vdots$ | | $\vdots$ | $b$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $y$ | $\vdots$ | | $\vdots$ |
| $\vdots$ | | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

Table 3.21: The profile for Step 7.

## Weakening Requirements

We have just learned that, if we wish to map any profile[10] of regular preferences to a transitive—hence regular—social preference relation satisfying a weak monotonicity requirement (Unanimity) and a weak ordinality requirement (IIA), we have to concentrate power in a problematically extreme way: in the hands of a dictator. Put more simply, preference aggregation involves trade-offs, as not all good things go together. We will now ruminate on how weakening some requirements might, in turn, weaken the subsequent negative consequences. The results raise an important point: power is *institutionally-determined.*

[10] By this I mean Universal Domain—that the domain of $f$ is all of $\mathscr{R}^n$, rather than some restricted set of "sensible" preference profiles.

### Some additional apparatus

In the immediate wake of Arrow's Theorem, many social scientists—notable among them James Buchanan, writing at FSU in much of this time period—wondered why one ought to demand transitivity (and hence regularity) of a social preference. Given how hard we leaned on Transitivity in the proof of Arrow's theorem, one natural question is: if we demand slightly less consistency in the societal preference, can we avoid dictatorship problems?

Let's consider two potential ways to weaken Transitivity:

1. we say a social preference $\mathbf{R}$ satisfies Quasi-Transitivity if its strict part $\mathbf{P}$ is transitive: that is, for $x, y, z \in X$ satisfying $x\mathbf{P}y$ and $y\mathbf{P}z$, we have $x\mathbf{P}z$.

2. we say a social preference **R** satisfies Acyclicity if it satisfies

$$x\mathbf{P}y\mathbf{P}z\mathbf{P}\cdots\mathbf{P}u\mathbf{P}v \implies x\mathbf{R}v.$$

We have the following:[11]

Transitivity $\implies$ Quasi-Transitivity $\implies$ Acyclicity.

[11] You'll prove this one in your problem set, but take my word for now.

In other words, Transitivity is *stronger* than Quasi-Transitivity, which in turn is stronger than Acyclicity. Frankly, Acyclicity is about as *weak* of a consistency requirement as you could make.

We need to develop a formalism for coalitions of individuals that can ensure a given preference outcome in the aggregate. We will do so with the following three definitions.

**3.4 Definition** *A set $L \subseteq N$ is* semidecisive for $x$ against $y$ *(written $x\tilde{D}_L y$) if, for every $R \in \mathscr{R}^n$,*

$$\left[x \succ_i y \text{ for all } i \in L \text{ and } y \succ_j x \text{ for all } j \notin L\right] \implies x\boldsymbol{P}y.$$

*A set $L \subseteq N$ is* decisive for $x$ against $y$ *(written $xD_L y$) if, for every $R \in \mathscr{R}^n$,*

$$[x \succ_i y \text{ for all } i \in L] \implies x\boldsymbol{P}y.$$

*A set $L \subseteq N$ is* decisive *if, for all ordered pairs $(x, y) \in X \times X$, $L$ is decisive for $x$ against $y$.*

Notice what is going on here. For semidecisiveness for $x$ against $y$ to hold, we need *all* members of a set $L$ to have $x \succ_i y$ and *all* members outside that set to have $y \succ_i x$. So, if a single individual is indifferent between $x$ and $y$, semidecisiveness for $x$ against $y$ tells us nothing. This is where decisiveness for $x$ against $y$ is of use: if we know $x \succ_i y$ for all members of a decisive (for $x$ against $y$) group $L$, then we know $x\mathbf{P}y$ in the aggregate, no matter how the people outside $L$ feel. Thus, decisiveness for $x$ against $y$ implies semidecisiveness for $x$ against $y$, but not vice versa. Finally, decisiveness generalizes decisiveness for $x$ against $y$ to the whole of the set of alternatives.[12]

[12] Sometimes you see other Arrovian axioms used to help get at this idea. Arrovian Neutrality demands that all alternatives be treated the same: if a group is enough to get $x$ preferred over $y$, then so too is it enough to get any $a$ over any $b$. We will prove this is true with our weaker assumptions in a moment.

To give you a feel for how these work, here are three examples.

1. Suppose $f$ is dictatorial, and let $i \in N$ be the dictator. Then a coalition $L \subseteq N$ is decisive if and only if $i \in L$.

2. Suppose $f$ is majority rule. Then a coalition $L \subseteq N$ is decisive if and only if $|L| > \frac{n}{2}$.

3. Suppose $f$ is the Pareto extension rule. Then a coalition $L \subseteq N$ is decisive if and only if it is the grand coalition itself, $N$.

Now that we have this notation in place, our next step is to show that, under the assumptions we have made, we can take the claim [$L$ is semidecisive for $x$ against $y$] and turn it into the claim [$L$ is decisive for $v$ against $w$ for any ordered pair $(v, w) \in X$]. In other words, we will evaluate whether we have the Arrovian Neutrality discussed in the footnote above.

**3.5 Lemma** *Let $f$ be a quasi-transitive preference aggregation rule that satisfies IIA and Unanimity. Then, if some set $L \subseteq N$ is semidecisive for $x$ against $y$ for some $x, y \in X$, then it is decisive for $v$ against $w$ for all ordered pairs $(v, w) \in X \times X$.*

This one is a little tricky, if only because its structure is a little less obvious.

*Proof*. We will work directly, so suppose the assumptions of the lemma are met, and let $L$ be semidecisive for $x$ against $y$. We work in two steps, with a third step synthesizing the two serving as a coda.

*Step 1*. Consider any profile $R'|_{\{x,y,z\}} \in \mathscr{R}^n$ satisfying

$$x \succ'_i y \succ'_i z \text{ for all } i \in L,$$
$$y \succ'_j x \text{ and } y \succ'_j z \text{ for all } j \notin L,$$

where $z \notin \{x, y\}$. (Throughout, $X$ has at least three alternatives, so this is fine.) Now, since $L$ is semidecisive for $x$ over $y$, it follows that we must have $x\mathbf{P}'y$. And, since all individuals $k$ have $y \succ'_k z$, Unanimity implies that $y\mathbf{P}'z$. What's more, since our preference aggregation rule is quasi-transitive, we may infer that $x\mathbf{P}'z$, too.

You will notice that, in the profile above, we have only specified a preference between $x$ and $z$ for individuals in the set $L$. For any other profile $R|_{\{x,z\}}$, we must also have $x \succ_i z$ for all $i \in L$, so IIA requires $x\mathbf{P}z$ for any other profile of this type. Thus, if $L$ is semidecisive for $x$ against $y$, then it is decisive for $x$ against any $z \notin \{x, y\}$. We write this as

$$\text{for all } z \notin \{x, y\},\ x\tilde{D}_L y \Longrightarrow xD_L z. \qquad (*)$$

Since $L$ is decisive for $x$ against $z$, it is semidecisive for $x$ against $z$, as well. This means we can use the exact same argument again, of the form

$$\begin{aligned} x\tilde{D}_L y &\Longrightarrow xD_L z, \\ &\Longrightarrow x\tilde{D}_L z, \\ &\Longrightarrow xD_L y, \end{aligned}$$

where the last line comes by switching $y$ and $z$ in the first part of the step.[13]

[13] In other words, repeat the first line, this time using it to infer that $x\tilde{D}_L z \Longrightarrow xD_L y$.

*Step 2.* Now let $R^\circ \in \mathscr{R}^n$ be any profile with $y \succ_i^\circ z$ for all $i \in L$, and consider another profile $R^+ \in \mathscr{R}^n$ such that

$$y \succ_i^+ x \succ_i^+ z \text{ for all } i \in L,$$
$$z \succ_j^+ x \text{ and } y \succ_i^+ x \text{ for all } j \notin L.$$

From the previous step, we know that $L$ is semidecisive for $x$ over $z$, so the preference aggregation rule must yield $x\mathbf{P}z$. Notice for this profile that all individuals $k$ have $y \succ_k^+ x$, so Unanimity requires that $y\mathbf{P}x$, too. And again, quasi-transitivity requires that $y\mathbf{P}z$. Once again, we have only specified preferences between $y$ and $z$ for individuals in $L$, so by the same IIA argument in Step 1 equating $R^+|_{\{y,z\}}$ to $R^\circ|_{\{y,z\}}$, we have that $L$ is decisive for $y$ over $z$. We arrive at the conclusion

$$\text{for all } z \notin \{x,y\},\ x\tilde{D}_L y \Longrightarrow yD_L z. \qquad (**)$$

As $L$ is decisive for $y$ against $z$, it is semidecisive for $y$ against $z$.

*Coda.* We just concluded that $L$ is semidecisive for $y$ against $z$. Using (*), we infer that $L$ is decisive for $y$ against $x$. Thus, for any pair of distinct alternatives $\{v,w\} \subseteq X \setminus \{x,y\}$,

$$x\tilde{D}_L y \Longrightarrow xD_L v \text{ by } (*),$$
$$\Longrightarrow vD_L w \text{ by } (**),$$

where the second step proceeded by replacing $y$ with $v$. □

We need only a final new piece of apparatus: for any ordered pair of alternatives $(a,b) \in X^2$, define $\lambda(a,b)$ as the size of the smallest semidecisive coalition for $a$ against $b$. And, let

$$\lambda = \min\left\{\lambda(a,b) \;\middle|\; (a,b) \in X^2\right\},$$

which is just the size of the smallest semidecisive set for any pair of alternatives.

## *Collective choice under quasi-transitivity*

Consider the Pareto extension rule, where $x\mathbf{P}y$ if and only if $x \succsim_i y$ for all $i \in N$ and there exists at least one $j \in N$ such that $x \succ_j y$. Clearly, this rule satisfies No Dictators in the extreme, since unanimity is required to generate $x\mathbf{P}y$. It also satisfies Unanimity by its very definition. And, it satisfies IIA in a particularly vacuous way: since it picks $x\mathbf{P}y$ only when $x \succsim_i y$ for all $i$ and $x \succ_j y$ for some $j$, there aren't very many other similar profiles, since the profile in question is quite restrictive.

It is also true that the Pareto extension rule is quasi-transitive. Suppose $x\mathbf{P}y$ and $y\mathbf{P}z$. From this we can infer that $x \gtrsim_i y$ and $y \gtrsim_i z$ for all $i$, and since the individual preference relations are transitive, we also have $x \gtrsim_i z$ for all $i$. Now, for some $j$ we have $x \succ_j y$. For this $j$, we have $x \succ_j y \gtrsim_j z$, which implies $x \succ_j z$. Since $x \gtrsim_i z$ for all $i$ and $x \succ_j z$ for some $z$, we infer that $x\mathbf{P}z$, demonstrating quasi-transitivity.

So, in other words, we have a rule that simultanously satisfies universal domain, No Dictators, Unanimity, IIA, and Quasi-Transitivity. That's not bad, right? But, you'll note that to do so, we have selected a rule that isn't very useful. As an extreme example, if $|N| \geqslant |X|$ and, for every $x \in X$, there exists some $i$ that has $x \succ_i y$ for all other $y \in X$, then the Pareto extension rule essentially says that all alternatives are equally good. The mechanism at work is that of a veto: the individual that likes $x$ best has a veto for $x$ over every other $y$, so that, for society, $x$ can't do any worse than $x\mathbf{I}y$.[14]

[14] Again, that need not be such a bad thing—quite a bit of ink has been written on, say, the Federalist Papers and how they conceive of prolific **I** outcomes.

It turns out this mechanism of vetos generalizes. So, let's formalize it.

**3.6 Definition** *An individual $i \in N$ has a* veto for $x$ against $y$ *if, for every $R \in \mathscr{R}^n$, $x \succ_i y$ implies $\neg(y\mathbf{P}x)$. Individual $i$ has* full veto power *if, for all ordered pairs $(x, y) \in X \times X$, $i$ has a veto for $x$ against $y$. Individual $i$ has a* semiveto for $x$ against $y$ *if, whenever $x \succ_i y$ and $x \sim_j y$ for all $j \neq i$, we have $x\mathbf{P}y$.*

A quick truism you'll prove later: if $i$ has a semiveto for $x$ against $y$, then she has a veto for any $a$ against any $b$.

From here, we can define a property of aggregation rules that makes some sense.

**3.7 Definition** *A preference aggregation rule $f : \mathscr{R}^n \to \mathscr{B}$ is* oligarchic *if there exists some $L \subseteq N$ (called the* oligarchy*) such that every member of $L$ has full veto power and $L$ is decisive.*

So, a dictator is a special case of an oligarchy with $|L| = 1$. The Pareto extension rule creates an oligarchy with $L = N$. There can be oligarchies of intermediate size, too. But, regardless, quasi-transitivity induces oligarchies.

**3.8 Proposition** *Let $f : \mathscr{R}^n \to \mathscr{B}$ be a quasi-transitive preference aggregation rule satisfying Unanimity and IIA. Then $f$ is oligarchic.*

*Proof.* Since $f$ satisfies Unanimity,[15] there exists a smallest semidecisive set for each pair $a$ and $b$. Let $\lambda(a, b)$ be the smallest semidecisive set for $a$ against $b$, and let $\lambda$ be the minimum value of $\lambda$. Now choose and fix $x$ and $y$ so that $\lambda(x, y) = \lambda$, and let $L$ be the smallest semidecisive set for $x$ against $y$. By Lemma 3.5, $L$ is decisive for all ordered

[15] This means that, at the very least, $0 < \lambda(a, b) \leqslant N$ for any $(a, b)$. Often the existence of the bounds matters more than the bandwidth itself.

pairs. If $\lambda = 1$, we are already done (dictatorships being oligarchies of size one that vacuously satisfy the veto requirement), so suppose $\lambda > 1$.

Now consider any profile $R$ such that $R|_{\{x,y,z\}}$ satisfies

$$x \succ_i y \succ_i z \text{ for some } i \in L,$$
$$z \succ_j x \succ_j y \text{ for all other } j \in L \setminus \{i\},$$
$$y \succ_k z \succ_k x \text{ for all } k \notin L.$$

Since $L$ is decisive for $x$ over $y$, we have $x\mathbf{P}y$. We cannot have $z\mathbf{P}y$ (else $L \setminus \{i\}$ would be semidecisive for $z$ over $y$, and $|L \setminus \{i\}| = \lambda - 1$, violating the premise that $\lambda$ is the size of the smallest semidecisive set), so $y\mathbf{R}z$. Quasi-Transitivity now kicks in, and we infer $x\mathbf{R}z$.[16] This means individual $i$ has a semiveto for $x$ against $z$ and thus has a veto over all ordered pairs $(a, b) \in X \times X$. Since $i$ was chosen arbitrarily within $L$, the argument holds for any $i \in L$; thus, $L$ is an oligarchy. We are done. □

[16] If $x\mathbf{R}z$ were not true, we would have $z\mathbf{P}x$; Quasi-Transitivity and the chain $z\mathbf{P}x\mathbf{P}y$ would yield $z\mathbf{P}y$, contradicting $y\mathbf{R}z$.

So, in relaxing transitivity down to quasi-transitivity, we concentrate power in a less aggressive way. However, the more we disaggregate a dictator's power into larger and larger oligarchies, the less decisive the decision rule becomes (in the sense that more indifferences emerge).[17]

[17] In other words, you can get "larger" oligarchies, including the oligarchy of size $n$, if you're willing to put up with less "crisp" societal decisions.

### *Acyclicity*

OK, so quasi-transitivity helps, but it doesn't really solve the problem of concentrated power. Maybe we are still asking too much. Let's see what we can squeeze out of acyclicity.

Again, we introduce some machinery. Given a preference aggregation rule $f$, define $\mathcal{L}(f) \subseteq \mathcal{P}(N)$ as the family of decisive coalitions given $f$.

**3.9 Definition** *A family of coalitions $\mathcal{C} \subseteq \mathcal{P}(N)$ is*

1. monotonic *if $C \in \mathcal{C}$ and $N \supseteq C' \supset C$ implies that $C' \in \mathcal{C}$;*

2. proper *if $C \in \mathcal{C}$ implies $N \setminus C \notin \mathcal{C}$.*

In other words, monotonic families are those where additional members to extant coalition lead yield other extant coalitions. Proper families are those where any two constituent coalitions must have at least one individual in common.

An example might help here. Let $N = \{1, 2, 3\}$. Consider the family of coalitions

$$\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

This family is not monotonic: observe that $\{1,2\} \in \mathcal{C}$, but $\{1,2,3\} \notin \mathcal{C}$. It is proper, however: for $\{1,2\}$, the complement $\{3\} \notin \mathcal{C}$; for $\{1,3\}$, the complement $\{2\} \notin \mathcal{C}$; and for $\{2,3\}$, the complement $\{1\} \notin \mathcal{C}$.

Now consider

$$\mathcal{C} = \{\{1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

This family is monotonic. Begin from $\{1\}$: every possible doubleton superset—namely $\{1,2\}$ and $\{1,3\}$—is in $\mathcal{C}$. And, for evety doubleton subset, the set $\{1,2,3\}$ is also included. In other words: for every element of the family, adding elements doesn't take you out of the family. But, this set is not proper: observe that $\{2,3\} \in \mathcal{C}$, but its complement $\{1\}$ is also an element of $C$.

We introduced these for a reason.

**3.10 Lemma** *For any aggregation rule $f$, $\mathcal{L}(f)$ is monotonic and proper.*

*Proof*. We begin with properness. Consider some $C \in \mathcal{C}$ that is decisive for some $x$ over some $y$. Define $\overline{C} = N \setminus C$. Now suppose the proposition is false, so that $\overline{C} \in \mathcal{C}$. Since $C$ is decisive for $x$ over $y$, it is semidecisive, so that if $x \succ_i y$ for all $i \in C$ and $y \succ_j x$ for all $j \in \overline{C}$, $x\mathbf{P}y$. But then $\overline{C}$ cannot be decisive for $x$ over $y$ (since $\mathbf{P}$ is asymmetric), contradicting our premise.

Now monotonicity. Consider an $L'$ such that, for all $i \in L'$, $x \succ_i y$. Trivially, this means that for any $L \subset L'$, $x \succ_j y$ for all $j \in L$. Thus, if $L \in \mathcal{L}(f)$, then $x\mathbf{P}y$. We then extend this to [$x \succ_i y$ for all $i \in L'$] implying $x\mathbf{P}y$ (remember that we are looking at decisive sets, not just semidecisive ones). Since this holds for any $x$ and $y$ (we chose them arbitrarily), it follows that $L'$ is itself a decisive set. □

We are now in position to discuss what kind of power concentration may follow from acyclic demands of social preferences.

**3.11 Definition** *A preference aggregation rule $f$ is* collegial *if and only if*

$$K(\mathcal{L}(f)) = \bigcap_{L \in \mathcal{L}(f)} L \neq \varnothing.$$

*The set $K(\mathcal{L}(f))$ is called the* collegium*.*

Any oligarchy is a collegium, but not all collegia are oligarchies. After all, the definition of a collegium has nothing to say about veto powers, nor about decisiveness. The collegium is just the set of individuals that are in all decisive sets. Obviously, if $i$ is a dictator, then the collegium is just $\{i\}$, too. And if $L$ is an oligarchy, then the collegium is a decisive set.

Here is what we can say.

**3.12 Proposition** *Let $|X| \geqslant n$. Then any acyclic aggregation rule satisfying Unanimity is collegial.*

*Proof*. Since $f$ satisfies Unanimity, we know there exists at least one decisive set, namely $N$. So, $\mathcal{L}(f) \neq \varnothing$.

We will work by contradiction. So, suppose that $f$ is not collegial. Then, for all $i \in N$ there exists some $L \in \mathcal{L}(f)$ such that $i \notin L$. (This means that the intersection of all the $L$ sets in $\mathcal{L}(f)$ will be empty, contradicting collegiality.) Now, Definition 3.9 tells us that $\mathcal{L}(f)$ is monotonic. Putting monotonicity and noncollegiality together, we infer that $N \setminus \{i\} \in \mathcal{L}(f)$ for all $i$.

Since $|X| \geqslant n$, we rearrange the outcomes in $X$ as $\{x_1, \ldots, x_n, \ldots, x_{|X|}\}$ (if $|X| = n$, this is just $\{x_1, \ldots, x_n\}$). Consider a profile $R \in \mathscr{R}^n$ with $R|_{\{x_1,\ldots,x_n\}}$ satisfying

$$
\begin{aligned}
x_1 \succ_1 x_2 \succ_1 \cdots \succ_1 x_n, \\
x_2 \succ_2 x_3 \succ_2 \cdots \succ_2 x_n \succ_2 x_1, \\
x_3 \succ_3 x_4 \succ_3 \cdots \succ_3 x_n \succ_3 x_1 \succ_3 x_2, \\
\vdots \\
x_n \succ_n x_1 \succ_n \cdots \succ_n x_{n-2} \succ_n x_{n-1}.
\end{aligned}
$$

Notice that, for all $k = 2, \ldots, n$, $P(x_{k-1}, x_k; R) = N \setminus \{k\}$. Since $N \setminus \{k\} \in \mathcal{L}(f)$ (see the paragraph above), we infer that $x_{k-1}\mathbf{P}x_k$. Note also that $P(x_n, x_1; R) = N \setminus \{1\}$ implying that $x_n\mathbf{P}x_1$. This creates the chain $x_n\mathbf{P}x_1\mathbf{P}x_2\mathbf{P}\cdots\mathbf{P}x_{n-1}\mathbf{P}x_n$, violating Acyclicity. □

## *Wrapping Up*

In this lecture, we discussed some of the rudiments of social choice. Arrow's Theorem taught us that no preference aggregation rule can simultaneously satisfy a weak notion of monotonicity, a weak notion of ordinality, and a relatively strong notion of regularity (namely Transitivity). So, we weakened Transitivity and found it easier to disperse power throughout the set of decision-makers.